

Crossed Bimodules over Rings and Shukla Cohomology

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Abstract

In this paper we present some applications of Ann-category theory to classification of crossed bimodules over rings, classification of ring extensions of the type of a crossed bimodule.

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1 Introduction

Crossed modules over groups were introduced by J. H. C. Whitehead [15]. A crossed module over a group G with kernel a G -module M represents an element in the cohomology $H^3(G, M)$ [8]. The results on group extensions of the type of a crossed module were also represented by the cohomology of groups [6].

Later, H-J. Baues [2] introduced crossed modules over \mathbf{k} -algebras. Crossed modules over \mathbf{k} -algebras which are \mathbf{k} -split with the same kernel M and cokernel B were classified by Hochschild cohomology $H_{Hoch}^3(B, M)$ [3].

In [4] the field \mathbf{k} is replaced by a commutative ring \mathbb{K} , and crossed modules over \mathbb{K} -algebras were called *crossed bimodules*. In particular, if $\mathbb{K} = \mathbb{Z}$ one obtains crossed bimodules over rings.

Crossed modules over groups can be defined over rings in a different way under the name of *E-systems*. The notion of an E-system is weaker than that of a crossed bimodule over rings.

Crossed modules over groups are often studied in the form of \mathcal{G} -groupoids [5], or strict 2-groups [1]. From this point, we represent E-systems in the form of strict Ann-categories (also called strict 2-rings). Hence, one can use the results on Ann-category theory to study crossed bimodules over rings.

The plan of this paper is, briefly, as follows. Section 2 is dedicated to review definitions and some basic facts concerning Ann-categories. In Section 3, we introduce the concept of an E-system and prove that there is an isomorphism between the category of regular E-systems and that of crossed bimodules over rings. The relation among these concepts and crossed \mathbf{C} -modules in the sense of T. Porter[10] is also discussed. The next section is devoted to showing a categorical equivalence of the category of E-systems and a subcategory of the category of strict Ann-categories, which is an extending of the result of R. Brown and C. Spencer [5].

The group extensions of the type of a crossed module were dealt with by R. Brown and O. Mucuk [6]. The similar results for ∂ -extensions by an algebra R were done by H-J. Baues and T. Pirashvili [4] in a particular case. In Section 5 we solve this problem for ring extensions of the type of an E-system by Shukla cohomology groups. Our classification result contains the result in [4] when R is a ring.

2 Ann-categories

We state a minimum of necessary concepts and facts of Ann-categories and Ann-functors (see [11]).

A *Gr-category* (or a *categorical group*) is a monoidal category in which all objects are invertible and the background category is a groupoid. A *Picard category* (or a *symmetric categorical group*) is a Gr-category equipped with a symmetry constraint which is compatible with associativity constraint.

Definition 1. An *Ann-category* consists of

- i) a category \mathcal{A} together with two bifunctors $\oplus, \otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$;
- ii) a fixed object $0 \in \text{Ob}(\mathcal{A})$ together with natural isomorphisms $\mathbf{a}_+, \mathbf{c}, \mathbf{g}, \mathbf{d}$ such that $(\mathcal{A}, \oplus, \mathbf{a}_+, \mathbf{c}, (0, \mathbf{g}, \mathbf{d}))$ is a Picard category;
- iii) a fixed object $1 \in \text{Ob}(\mathcal{A})$ together with natural isomorphisms $\mathbf{a}, \mathbf{l}, \mathbf{r}$ such that $(\mathcal{A}, \otimes, \mathbf{a}, (1, \mathbf{l}, \mathbf{r}))$ is a monoidal category;
- iv) natural isomorphisms $\mathfrak{L}, \mathfrak{R}$ given by

$$\begin{aligned} \mathfrak{L}_{A,X,Y} : A \otimes (X \oplus Y) &\longrightarrow (A \otimes X) \oplus (A \otimes Y), \\ \mathfrak{R}_{X,Y,A} : (X \oplus Y) \otimes A &\longrightarrow (X \otimes A) \oplus (Y \otimes A) \end{aligned}$$

such that the following conditions hold:

(Ann - 1) for $A \in \text{Ob}(\mathcal{A})$, the pairs $(L^A, \check{L}^A), (R^A, \check{R}^A)$ defined by

$$\begin{aligned} L^A &= A \otimes - & R^A &= - \otimes A \\ \check{L}_{X,Y}^A &= \mathfrak{L}_{A,X,Y} & \check{R}_{X,Y}^A &= \mathfrak{R}_{X,Y,A} \end{aligned}$$

are \oplus -functors which are compatible with \mathbf{a}_+ and \mathbf{c} ;

(Ann - 2) for all $A, B, X, Y \in \text{Ob}(\mathcal{A})$, the following diagrams commute

$$\begin{array}{ccc} (AB)(X \oplus Y) & \xleftarrow{\mathbf{a}_{A,B,X \oplus Y}} A(B(X \oplus Y)) & \xrightarrow{id_A \otimes \check{L}^B} A(BX \oplus BY) \\ \downarrow \check{L}^{AB} & & \downarrow \check{L}^A \\ (AB)X \oplus (AB)Y & \xleftarrow{\mathbf{a}_{A,B,X} \oplus \mathbf{a}_{A,B,Y}} & A(BX) \oplus A(BY) \end{array}$$

$$\begin{array}{ccc} (X \oplus Y)(BA) & \xrightarrow{\mathbf{a}_{X \oplus Y,B,A}} ((X \oplus Y)B)A & \xrightarrow{\check{R}^B \otimes id_A} (XB \oplus YB)A \\ \downarrow \check{R}^{BA} & & \downarrow \check{R}^A \\ X(BA) \oplus Y(BA) & \xrightarrow{\mathbf{a}_{X,B,A} \oplus \mathbf{a}_{Y,B,A}} & (XB)A \oplus (YB)A \end{array}$$

$$\begin{array}{ccccc}
(A(X \oplus Y)B) & \xleftarrow{\mathbf{a}_{A,X \oplus Y,B}} & A((X \oplus Y)B) & \xrightarrow{id_A \otimes \check{R}^B} & A(XB \oplus YB) \\
\downarrow \check{L}^A \otimes id_B & & & & \downarrow \check{L}^A \\
(AX \oplus AY)B & \xrightarrow{\check{R}^B} & (AX)B \oplus (AY)B & \xleftarrow{\mathbf{a} \oplus \mathbf{a}} & A(XB) \oplus A(YB) \\
\downarrow \check{R}^X \oplus \check{R}^Y & & & & \downarrow \check{L}^A \oplus \check{L}^B \\
(A \oplus B)X \oplus (A \oplus B)Y & \xleftarrow{\check{L}^{A \oplus B}} & (A \oplus B)(X \oplus Y) & \xrightarrow{\check{R}^{X \oplus Y}} & A(X \oplus Y) \oplus B(X \oplus Y) \\
\downarrow \check{R}^X \oplus \check{R}^Y & & & & \downarrow \check{L}^A \oplus \check{L}^B \\
(AX \oplus BX) \oplus (AY \oplus BY) & \xrightarrow{\mathbf{v}} & (AX \oplus AY) \oplus (BX \oplus BY) & &
\end{array}$$

where $\mathbf{v} = \mathbf{v}_{U,V,Z,T} : (U \oplus V) \oplus (Z \oplus T) \longrightarrow (U \oplus Z) \oplus (V \oplus T)$ is a unique morphism constructed from $\oplus, \mathbf{a}_+, \mathbf{c}, id$ of the symmetric monoidal category (\mathcal{A}, \oplus) ;

(Ann - 3) for the unit $1 \in \text{Ob}(\mathcal{A})$ of the operation \otimes , the following diagrams commute

$$\begin{array}{ccc}
1(X \oplus Y) & \xrightarrow{\check{L}^1} & 1X \oplus 1Y \\
\searrow 1_{X \oplus Y} & & \swarrow 1_X \oplus 1_Y \\
& X \oplus Y &
\end{array}
\quad
\begin{array}{ccc}
(X \oplus Y)1 & \xrightarrow{\check{R}^1} & X1 \oplus Y1 \\
\searrow r_{X \oplus Y} & & \swarrow r_X \oplus r_Y \\
& X \oplus Y &
\end{array}$$

An Ann-category \mathcal{A} is *regular* if its symmetry constraint satisfies the condition $\mathbf{c}_{X,X} = id$, and *strict* if all of its constraints are identities.

Example 1. Let $\mathcal{A} = (\mathcal{A}, \oplus)$ be a Picard category whose unity and associativity constraints are identities. Denote by $\text{End}(\mathcal{A})$ a category whose objects are symmetric monoidal functors from \mathcal{A} to \mathcal{A} and whose morphisms are \oplus -morphisms. Then, $\text{End}(\mathcal{A})$ is a Picard category together with the operation \oplus on monoidal functors and on morphisms. In this \oplus -category, the unity and associativity constraints are identities, the commutativity constraint is given by

$$(\mathbf{c}_{F,G}^*)_X = \mathbf{c}_{FX, GX}, \quad X \in \text{Ob}(\mathcal{A}), \quad F, G \in \text{End}(\mathcal{A}).$$

The operation \otimes on $\text{End}(\mathcal{A})$ is naturally defined being the composition of functors. Then, $\text{End}(\mathcal{A})$ together with two operations \oplus, \otimes is an Ann-category in which the left distributivity constraint is given by

$$(\mathfrak{L}_{F,G,H}^*)_X = \check{F}_{GX, HX}, \quad X \in \text{Ob}(\mathcal{A}),$$

and other constraints are identities (for details, see [12]).

Example 2. Let R be a ring with a unit and M be an R -bimodule. The pair $\mathcal{I} = (R, M)$ is a category whose objects are elements of R and whose morphisms are automorphisms $(r, a) : r \rightarrow r, r \in R, a \in M$. The composition of morphisms is given by the addition in M . Two operations \oplus and \otimes on \mathcal{I} is defined by

$$x \oplus y = x + y, \quad (x, a) + (y, b) = (x + y, a + b),$$

$$x \otimes y = xy, (x, a) \otimes (y, b) = (xy, xb + ay).$$

The constraints of \mathcal{I} are identities, except for left distributivity and commutativity constraints which are given by

$$\mathcal{L}_{x,y,z} = (\bullet, \lambda(x, y, z)) : x(y + z) \rightarrow xy + xz,$$

$$\mathbf{c}_{x,y}^+ = (\bullet, \eta(x, y)) : x + y \rightarrow y + x,$$

where $\lambda : R^3 \rightarrow M, \eta : R^2 \rightarrow M$ are functions satisfying the appropriate coherence conditions.

Here are standard consequences of the axioms of an Ann-category.

Lemma 1. *For every Ann-category \mathcal{A} there exist uniquely isomorphisms*

$$\hat{L}^A : A \otimes 0 \rightarrow 0, \quad \hat{R}^A : 0 \otimes A \rightarrow 0,$$

where $A \in \text{Ob}(\mathcal{A})$, such that \oplus -functors $(L^A, \check{L}, \hat{L}^A)$ and $(R^A, \check{R}, \hat{R}^A)$ are compatible with unit constraints $(0, \mathbf{g}, \mathbf{d})$.

It is easy to see that if (\mathcal{A}, \oplus) and (\mathcal{A}', \oplus) are Gr-categories, then every \oplus -functor $(F, \check{F}) : \mathcal{A} \rightarrow \mathcal{A}'$, which is compatible with associativity constraints, is a monoidal functor. Thus, we state the following definition.

Definition 2. Let \mathcal{A} and \mathcal{A}' be Ann-categories. An *Ann-functor* $(F, \check{F}, \tilde{F}, F_*) : \mathcal{A} \rightarrow \mathcal{A}'$ consists of a functor $F : \mathcal{A} \rightarrow \mathcal{A}'$, natural isomorphisms

$$\check{F}_{X,Y} : F(X \oplus Y) \rightarrow F(X) \oplus F(Y), \quad \tilde{F}_{X,Y} : F(X \otimes Y) \rightarrow F(X) \otimes F(Y),$$

and an isomorphism $F_* : F(1) \rightarrow 1'$ such that (F, \check{F}) is a symmetric monoidal functor for the operation \oplus , (F, \tilde{F}, F_*) is a monoidal functor for the operation \otimes , and the following diagrams commute

$$\begin{array}{ccccc} F(X(Y \oplus Z)) & \xrightarrow{\check{F}} & FX.F(Y \oplus Z) & \xrightarrow{id \otimes \check{F}} & FX(FY \oplus FZ) \\ \downarrow F(\mathfrak{L}) & & & & \downarrow \mathfrak{L}' \\ F(XY \oplus XZ) & \xrightarrow{\check{F}} & F(XY) \oplus F(XZ) & \xrightarrow{\check{F} \oplus \check{F}} & FX.FY \oplus FX.FZ \\ \\ F((X \oplus Y)Z) & \xrightarrow{\check{F}} & F(X \oplus Y).FZ & \xrightarrow{\check{F} \otimes id} & (FX \oplus FY)FZ \\ \downarrow F(\mathfrak{R}) & & & & \downarrow \mathfrak{R}' \\ F(XZ \oplus YZ) & \xrightarrow{\check{F}} & F(XZ) \oplus F(YZ) & \xrightarrow{\check{F} \oplus \check{F}} & FX.FZ \oplus FY.FZ. \end{array}$$

These diagrams are called the *compatibility* of the functor F with the distributivity constraints.

An *Ann-morphism* (or a *homotopy*)

$$\theta : (F, \check{F}, \tilde{F}, F_*) \rightarrow (F', \check{F}', \tilde{F}', F'_*)$$

between Ann-functors is an \oplus -morphism, as well as an \otimes -morphism.

If there exist an Ann-functor $(F', \check{F}', \hat{F}', F'_*) : \mathcal{A}' \rightarrow \mathcal{A}$ and Ann-morphisms $F'F \xrightarrow{\sim} id_{\mathcal{A}}$, $FF' \xrightarrow{\sim} id_{\mathcal{A}'}$, we say that $(F, \check{F}, \hat{F}, F_*)$ is an *Ann-equivalence*, and $\mathcal{A}, \mathcal{A}'$ are *Ann-equivalent*.

For an Ann-category \mathcal{A} , the set $R = \pi_0 \mathcal{A}$ of isomorphism classes of the objects in \mathcal{A} is a ring with two operations $+$, \times induced by the functors \oplus , \otimes on \mathcal{A} , and the set $M = \pi_1 \mathcal{A} = \text{Aut}(0)$ is a group with the composition denoted by $+$. Moreover, M is a R -bimodule with the actions

$$sa = \lambda_X(a), \quad as = \rho_X(a),$$

for $X \in s, s \in \pi_0 \mathcal{A}, a \in \pi_1 \mathcal{A}$, and λ_X, ρ_X satisfy

$$\begin{aligned} \lambda_X(a) \circ \hat{L}^X &= \hat{L}^X \circ (id \otimes \mathbf{a}) : X.0 \rightarrow 0, \\ \rho_X(a) \circ \hat{R}^X &= \hat{R}^X \circ (\mathbf{a} \otimes id) : 0.X \rightarrow 0. \end{aligned}$$

We recall briefly some main facts of the construction of the reduced Ann-category $S_{\mathcal{A}}$ of \mathcal{A} via the structure transport (for details, see [11]). The objects of $S_{\mathcal{A}}$ are the elements of the ring $\pi_0 \mathcal{A}$. A morphism is an automorphism $(s, a) : s \rightarrow s, s \in \pi_0 \mathcal{A}, a \in \pi_1 \mathcal{A}$. The composition of morphisms is given by

$$(s, a) \circ (s, b) = (s, a + b).$$

For each $s \in \pi_0 \mathcal{A}$, choose an object $X_s \in \text{Ob}(\mathcal{A})$ such that $X_0 = 0, X_1 = 1$, and choose an isomorphism $i_X : X \rightarrow X_s$ such that $i_{X_s} = id_{X_s}$. We obtain two functors

$$\begin{cases} G : \mathcal{A} \rightarrow S_{\mathcal{A}} \\ G(X) = [X] = s \\ G(X \xrightarrow{f} Y) = (s, \gamma_{X_s}^{-1}(i_Y f i_X^{-1})), \end{cases} \quad \begin{cases} H : S_{\mathcal{A}} \rightarrow \mathcal{A} \\ H(s) = X_s \\ H(s, u) = \gamma_{X_s}(u), \end{cases}$$

for $X, Y \in s, f : X \rightarrow Y$, and γ_X

$$\gamma_X(\mathbf{a}) = \mathbf{g}_X \circ (\mathbf{a} \oplus id) \circ \mathbf{g}_X^{-1}. \quad (1)$$

Two operations on $S_{\mathcal{A}}$ are given by

$$\begin{aligned} s \oplus t &= G(H(s) \oplus H(t)) = s + t, \\ s \otimes t &= G(H(s) \otimes H(t)) = st, \\ (s, a) \oplus (t, b) &= G(H(s, a) \oplus H(t, b)) = (s + t, a + b), \\ (s, a) \otimes (t, b) &= G(H(s, a) \otimes H(t, b)) = (st, sb + at), \end{aligned}$$

for $s, t \in \pi_0 \mathcal{A}, a, b \in \pi_1 \mathcal{A}$. Clearly, they do not depend on the choice of the representative (X_s, i_X) .

The constraints in $S_{\mathcal{A}}$ are defined by sticks. A *stick* of \mathcal{A} is a representative (X_s, i_X) such that

$$\begin{aligned} i_{0 \oplus X_t} &= \mathbf{g}_{X_t}, & i_{X_s \oplus 0} &= \mathbf{d}_{X_s}, \\ i_{1 \otimes X_t} &= \mathbf{l}_{X_t}, & i_{X_s \otimes 1} &= \mathbf{r}_{X_s}, \quad i_{0 \otimes X_t} = \hat{R}^{X_t}, \quad i_{X_s \otimes 0} = \hat{L}^{X_s}. \end{aligned}$$

The unit constraints in $S_{\mathcal{A}}$ are $(0, id, id)$ and $(1, id, id)$. The family of the rest ones, $h = (\xi, \eta, \alpha, \lambda, \rho)$, is defined by the compatibility of the constraints $\mathbf{a}_+, \mathbf{c}, \mathbf{a}, \mathfrak{L}, \mathfrak{R}$ of \mathcal{A} with the functor H and isomorphisms

$$\check{H} = i_{X_s \oplus X_t}^{-1}, \tilde{H} = i_{X_s \otimes X_t}^{-1}. \quad (2)$$

Then $(H, \check{H}, \tilde{H}) : S_{\mathcal{A}} \rightarrow \mathcal{A}$ is an Ann-equivalence. Besides, the functor $G : \mathcal{A} \rightarrow S_{\mathcal{A}}$ together with isomorphisms

$$\check{G}_{X,Y} = G(i_X \oplus i_Y), \quad \tilde{G}_{X,Y} = G(i_X \otimes i_Y) \quad (3)$$

is also an Ann-equivalence. We refer to $S_{\mathcal{A}}$ as an Ann-category of *type* (R, M) , and $(H, \check{H}, \tilde{H}), (G, \check{G}, \tilde{G})$ are *canonical* Ann-equivalences. The family of constraints $h = (\xi, \eta, \alpha, \lambda, \rho)$ of $S_{\mathcal{A}}$ is called a *structure* of the Ann-category of type (R, M) .

Mac Lane [7] and Shukla [14] cohomomology groups at low dimensions are used to classify Ann-categories and regular Ann-categories, respectively. A structure h of the Ann-category $S_{\mathcal{A}}$ is an element in the group of Mac Lane 3-cocycles $Z_{MacL}^3(R, M)$. In the case when \mathcal{A} is regular, $h \in Z_{Shu}^3(R, M)$.

Proposition 2 (Proposition 11 [11]). *Let \mathcal{A} and \mathcal{A}' be Ann-categories.*

i) *Every Ann-functor $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$ induces an Ann-functor $S_F : S_{\mathcal{A}} \rightarrow S_{\mathcal{A}'}$ of type (p, q) , where*

$$p = F_0 : \pi_0 \mathcal{A} \rightarrow \pi_0 \mathcal{A}', \quad [X] \mapsto [FX],$$

$$q = F_1 : \pi_1 \mathcal{A} \rightarrow \pi_1 \mathcal{A}', \quad u \mapsto \gamma_{F_0}^{-1}(Fu),$$

for γ is a map given by the relation (1).

ii) *F is an equivalence if and only if F_0, F_1 are isomorphisms.*

iii) *The Ann-functor S_F satisfies*

$$S_F = G' \circ F \circ H,$$

where H, G' are canonical Ann-equivalences.

Let $\mathcal{S} = (R, M, h), \mathcal{S}' = (R', M', h')$ be Ann-categories. Since $\check{F}_{x,y} = (\bullet, \tau(x, y)), \tilde{F}_{x,y} = (\bullet, \nu(x, y))$, then $g_F = (\tau, \nu)$ is a pair of maps *associated* with (\check{F}, \tilde{F}) , we thus can regard an Ann-functor $F : \mathcal{S} \rightarrow \mathcal{S}'$ as a triple (p, q, g_F) . It follows from the compatibility of F with the constraints that

$$q_* h - p^* h' = \partial(g_F),$$

where q_*, p^* are canonical homomorphisms,

$$Z_{MacL}^3(R, M) \xrightarrow{q_*} Z_{MacL}^3(R, M') \xleftarrow{p^*} Z_{MacL}^3(R', M').$$

Further, two Ann-functors $(F, g_F), (F', g_{F'})$ are homotopic if and only if $F = F'$, that is, they are the same type of (p, q) , and there exists a function $t : R \rightarrow M'$ such that $g_{F'} = g_F + \partial t$.

We denote by

$$\text{Hom}_{(p,q)}^{\text{Ann}}[\mathcal{S}, \mathcal{S}']$$

the set of homotopy classes of Ann-functors of type (p, q) from \mathcal{S} to \mathcal{S}' .

Let $F : \mathcal{S} \rightarrow \mathcal{S}'$ be an Ann-functor of type (p, q) , then the function

$$k = q_*h - p^*h' \in Z_{\text{MacL}}^3(R, M') \quad (4)$$

is called an *obstruction* of F .

Theorem 3 (Theorem 4.4, 4.5 [13]). *A functor $F : \mathcal{S} \rightarrow \mathcal{S}'$ of type (p, q) is an Ann-functor if and only if its obstruction \bar{k} vanishes in $H_{\text{MacL}}^3(R, M')$. Then, there exists a bijection*

$$\text{Hom}_{(p,q)}^{\text{Ann}}[\mathcal{S}, \mathcal{S}'] \leftrightarrow H_{\text{MacL}}^2(R, M') (= H_{\text{Shu}}^2(R, M')).$$

3 Crossed bimodules over rings and regular E-systems

The results on crossed bimodules can be found in [2, 3, 4, 9]. We shall show a characteristic of crossed bimodules when the base ring \mathbb{K} is the ring of integers \mathbb{Z} . Based on this characteristic, we can establish the relation between crossed bimodules over rings and Ann-category theory in the next section.

Definition [9]. A *crossed bimodule* is a triple (B, D, d) , where D is an associative \mathbb{K} -algebra, B is a D -bimodule and $d : B \rightarrow D$ is a homomorphism of D -bimodules such that

$$d(b)b' = bd(b'), \quad b, b' \in B. \quad (5)$$

A *morphism* $(k_1, k_0) : (B, D, d) \rightarrow (B', D', d')$ of crossed bimodules is a pair $k_1 : B \rightarrow B'$, $k_0 : D \rightarrow D'$, where k_1 is a group homomorphism, k_0 is a \mathbb{K} -algebra homomorphism such that

$$k_0d = d'k_1 \quad (6)$$

and

$$k_1(xb) = k_0(x)k_1(b), \quad k_1(bx) = k_1(b)k_0(x), \quad (7)$$

for all $x \in D, b \in B$.

The condition (7) shows that k_1 is a homomorphism of D -bimodules, where B' is a D -bimodule with the action $xb' = k_0(x)b'$, $b'x = b'k_0(x)$.

Below, the base ring \mathbb{K} is the ring of integers \mathbb{Z} , and a crossed bimodule (B, D, d) is called a *crossed bimodule over rings*. Thus, D is a ring with unit.

In order to introduce the concept of an E-system, we now recall some terminologies due to Mac Lane [7]. The set of all bimultiplications of a ring A is a ring denoted by M_A . For each element c of A , a bimultiplication μ_c is defined by

$$\mu_c a = ca, a\mu_c = ac, a \in A$$

we call μ_c an *inner bimultiplication*. Then $C_A = \{c \in A \mid \mu_c = 0\}$ is called the *bicenter* of A .

The bimultiplications σ and τ are *permutable* if for every $a \in A$,

$$\sigma(a\tau) = (\sigma a)\tau, \quad \tau(a\sigma) = (\tau a)\sigma. \quad (8)$$

We now introduce the main concept of the present paper which can be seen as a version of the concept of a crossed module over rings.

Definition 3. An *E-system* is a quadruple (B, D, d, θ) , where $d : B \rightarrow D$, $\theta : D \rightarrow M_B$ are the ring homomorphisms such that the following diagram commutes

$$\begin{array}{ccc} B & \xrightarrow{d} & D \\ & \searrow \mu & \swarrow \theta \\ & M_B & \end{array} \quad (9)$$

and the following relations hold for all $x \in D, b \in B$,

$$d(\theta_x b) = x.d(b), \quad d(b\theta_x) = d(b).x. \quad (10)$$

An E-system (B, D, d, θ) is *regular* if θ is a 1-homomorphism (a homomorphism carries the identity to the identity), and the elements of $\theta(D)$ are permutable.

A *morphism* $(f_1, f_0) : (B, D, d, \theta) \rightarrow (B', D', d', \theta')$ of E-systems consists of ring homomorphisms $f_1 : B \rightarrow B'$, $f_0 : D \rightarrow D'$ such that

$$f_0 d = d' f_1 \quad (11)$$

and f_1 is an *operator homomorphism*, that is,

$$f_1(\theta_x b) = \theta'_{f_0(x)} f_1(b), \quad f_1(b\theta_x) = f_1(b)\theta'_{f_0(x)}. \quad (12)$$

In this paper, an E-system (B, D, d, θ) is sometimes denoted by $B \xrightarrow{d} D$, or $B \rightarrow D$.

Example 3. If B is a two-sided ideal in D , then (B, D, d, θ) is a regular E-system, where d is an inclusion, $\theta : D \rightarrow M_B$ is given by the *bimultiplication type*, that is,

$$\theta_x b = xb, \quad b\theta_x = bx, \quad x \in D, b \in B.$$

Example 4. Let D be a ring, B be a D -bimodule, $\mathbf{0} : B \rightarrow D$ is the zero homomorphism of D -bimodules. B can be considered as a ring with zero multiplication defined by $b.b' = \mathbf{0}(b)b' = b\mathbf{0}(b') = 0$, for all $b, b' \in B$. Then, $(B, D, \mathbf{0}, \theta)$ is a regular E-system, where θ is given by the action of D -bimodules.

Example 5. Let B be a ring, M_B be the ring of bimultiplications of B , and $\mu : B \rightarrow M_B$ be the homomorphism which carries an element b in B to an inner bimultiplication of B . Then (B, M_B, μ, id) is an E-system. In general, this E-system is not regular.

Standard consequences of the axioms of an E-system are as below.

Proposition 4. *Let (B, D, d, θ) be an E-system.*

- i) $\text{Kerd} \subset C_B$.
- ii) $\text{Im}d$ is an ideal in D .
- iii) The homomorphism θ induces a homomorphism $\varphi : D \rightarrow M_{\text{Kerd}}$ given by

$$\varphi_x = \theta_x|_{\text{Kerd}}.$$

- iv) Kerd is a Cokerd-bimodule with the actions

$$sa = \varphi_x(a), \quad as = (a)\varphi_x, \quad a \in \text{Kerd}, \quad x \in s \in \text{Cokerd}.$$

To state the relation between regular E-systems and crossed bimodules over rings, one recalls the following definition.

Definition 4. A functor $\Phi : \mathbf{C} \rightarrow \mathbf{C}'$ is an isomorphism of categories if it is bijective on objects and on morphism sets.

Theorem 5. *The categories of regular E-systems and of crossed bimodules over rings are isomorphic.*

Proof. Let ${}_sB = (B, D, d, \theta)$ be a regular E-system. The abelian additive group B is a D -bimodule with the actions

$$xb = \theta_x b, \quad bx = b\theta_x, \quad (13)$$

for $x \in D, b \in B$. It is then easy to check that the axioms of a crossed bimodule hold. For example, the relation (5) follows from the relation (9),

$$d(b)b' = \theta_{d(b)}(b') \stackrel{(9)}{=} \mu_b(b') = bb' = b\mu_{b'} \stackrel{(9)}{=} b\theta_{d(b')} = bd(b'),$$

since $\mu_b, \mu_{b'}$ are inner bimultiplications of the ring B . Besides, the regularity of the E-system (B, D, d, θ) is necessary and sufficient for the two-sided module B to be a D -bimodule.

Conversely, if ${}_cB = (B, D, d)$ is a crossed bimodule then B has a ring structure with the multiplication

$$b * b' := d(b)b' = bd(b'), \quad b, b' \in B. \quad (14)$$

Clearly, $d : B \rightarrow D$ is a ring homomorphism since for all $b, b' \in B$,

$$d(b * b') = d(d(b)b') = d(b)d(b').$$

The map $\theta : D \rightarrow M_B$ is defined by the D -bimodule actions (13). Then, θ is a homomorphism with image in M_B , the elements of $\theta(D)$ are permutable since B is a D -bimodule. The homomorphism θ satisfies the condition (10) since d is a homomorphism of bimodules. Thus, the correspondence ${}_sB \mapsto {}_cB$ is bijective on objects.

Now, if $(f_1, f_0) : (B, D, d, \theta) \rightarrow (B', D', d', \theta')$ is a morphism of E-systems, it is then clear that (f_1, f_0) satisfies the relation (6).

Further, for all $x \in D, b \in B$, one has

$$f_1(xb) \stackrel{(13)}{=} f_1(\theta_x b) \stackrel{(12)}{=} \theta'_{f_0(x)} f_1(b) \stackrel{(13)}{=} f_0(x) f_1(b) = x f_1(b).$$

Similarly, one obtains $f_1(bx) = f_1(b)x$. This means that the pair (f_1, f_0) is a morphism of crossed bimodules.

Conversely, let $(k_1, k_0) : (B, D, d) \rightarrow (B', D', d')$ be a morphism of crossed bimodules. We show that k_1 is a ring homomorphism. According to the determination of the multiplication on the ring B , we have

$$k_1(b * b') \stackrel{(14)}{=} k_1(d(b)b') \stackrel{(7)}{=} k_0(d(b))k_1(b') \stackrel{(6)}{=} d'(k_1(b))k_1(b') \stackrel{(14)}{=} k_1(b) * k_1(b'),$$

for all $b, b' \in B$. Besides, the pair (k_1, k_0) also satisfies (12). \square

By the above proposition, the notion of an E-system can be seen as a weakened version of the notion of a crossed bimodule over rings.

We now discuss the relationship among the above concepts and the concept of a crossed module of D -structures in the category \mathbf{C} of Ω -groups (see [10]). For convenience, such a crossed module is called a crossed \mathbf{C} -module. T. Porter proved that there is an equivalence between the category of crossed \mathbf{C} -modules and that of internal categories in \mathbf{C} . A crossed \mathbf{C} -module can be described as follows.

Proposition 6 (Proposition 2 [10]). *Given a D -structure on B , $d : B \rightarrow D$ is a crossed \mathbf{C} -module if and only if the following conditions are satisfied for all $b, b_1, b_2 \in B, x \in D, * \in \Omega'_2 \subset \Omega$*

- (i). $d((-x) \cdot b) = -x + d(b) + x$,
- (ii). $(-d(b_1)) \cdot b_2 = -b_1 + b_2 + b_1$,
- (iii). $d(b_1) * b_2 = b_1 * b_2 = b_1 * d(b_2)$,
- (iv). $\begin{cases} d(xb) = x * d(b) \\ d(bx) = d(b) * x. \end{cases}$

Here $*$ is a binary operation which is not the group operation $+$, the actions $x \cdot b, x * b$ are given by

$$x \cdot b = s(x) + b - s(x),$$

$$x * b = s(x) * b,$$

where s is the morphism in the split exact sequence

$$0 \rightarrow B \xrightarrow{i} E \begin{matrix} \xrightarrow{p} \\ \xleftarrow{s} \end{matrix} D \rightarrow 0.$$

To establish the link between these crossed \mathbf{C} -modules and crossed modules over rings, we take \mathbf{C} to be a category whose objects are rings. The morphisms of \mathbf{C} are ring homomorphisms which are not necessarily 1-homomorphisms.

Proposition 7. *Every crossed \mathbf{C} -module is a crossed bimodule over rings.*

Proof. Let $d : B \rightarrow D$ be a crossed \mathbf{C} -module. Then d is a ring homomorphism, and D acts on B by

$$xb = s(x)b, \quad bx = bs(x), \quad x \in D, b \in B. \quad (15)$$

The map $\theta : D \rightarrow M_B$ is given by

$$\theta_x(b) = xb, \quad (b)\theta_x = bx.$$

Since s is a ring homomorphism, so is θ . The relation (9) follows from the condition (iii). Indeed, for $b, b' \in B$

$$(\theta d)(b)(b') = \theta_{db}(b') = (db)b' = bb' = \mu_b(b').$$

It follows from (iv) that $d(\theta_x(b)) = d(xb) = xd(b)b$. This means the relation (10) holds, and therefore (B, D, d, θ) is an E-system. \square

One can see that a crossed \mathbf{C} -module $d : B \rightarrow D$ satisfies most of the conditions of a crossed bimodule over rings. We first see that B is a D -bimodule with the action (15). By (iv), the ring homomorphism $d : B \rightarrow D$ is a D -bimodule. The relation (5) follows directly from the condition (iii). Note that the ring D is not necessarily unitary and if it has a unit, the ring B is not assumed to be a unitary D -bimodule. These investigations show that the concept of a crossed \mathbf{C} -module can be seen as a weakened version of the concept of a crossed bimodule over rings.

Remark. Since \mathbf{C} can be any of categories of Ω -groups, use of crossed \mathbf{C} -modules has resulted in various contexts. However, in each particular case there is a certain restriction. For example, by Proposition 3 [10] $\text{Ker} d$ is singular; while for crossed modules over groups, (or crossed modules over rings) $\text{Ker} d$ is a subgroup in the center (or the bicenter) of B .

Since rings with unit are not Ω -groups, one can not seek a relation among the category of crossed \mathbf{C} -modules, cohomology of algebras and cohomology of rings.

4 Strict Ann-categories and E-systems

Crossed modules over groups are often studied in the form of strict 2-groups (see [1, 5, 6]). In this section, we prove that E-systems and strict Ann-categories are equivalent.

For every E-system (B, D, d, θ) we can construct a strict Ann-category $\mathcal{A} = \mathcal{A}_{B \rightarrow D}$, called the Ann-category *associated to* the E-system (B, D, d, θ) , as follows. One sets

$$\text{Ob}(\mathcal{A}) = D,$$

and for two objects x, y of \mathcal{A} ,

$$\text{Hom}(x, y) = \{b \in B \mid y = d(b) + x\}.$$

The composition of morphisms is given by

$$(x \xrightarrow{b} y \xrightarrow{c} z) = (x \xrightarrow{b+c} z).$$

Two operations \oplus, \otimes on objects are given by the operations $+, \times$ on the ring D . For the morphisms, we set

$$(x \xrightarrow{b} y) \oplus (x' \xrightarrow{b'} y') = (x + x' \xrightarrow{b+b'} y + y'),$$

$$(x \xrightarrow{b} y) \otimes (x' \xrightarrow{b'} y') = (xx' \xrightarrow{bb' + b\theta_{x'} + \theta_x b'} yy').$$

Based on the definition of an E-system, it is easy to verify that \mathcal{A} is an Ann-category with the strict constraints.

Conversely, for every strict Ann-category $(\mathcal{A}, \oplus, \otimes)$ one can define an E-system $C_{\mathcal{A}} = (B, D, d, \theta)$. Indeed, let

$$D = \text{Ob}(\mathcal{A}), \quad B = \{0 \xrightarrow{b} x \mid x \in D\}.$$

Then, D is a ring with two operations

$$x + y = x \oplus y, \quad xy = x \otimes y,$$

and B is a ring with two operations

$$b + c = b \oplus c, \quad bc = b \otimes c.$$

The homomorphisms $d : B \rightarrow D$ and $\theta : D \rightarrow M_B$ are defined by

$$d(0 \xrightarrow{b} x) = x,$$

$$\theta_y(0 \xrightarrow{b} x) = (0 \xrightarrow{id_y \otimes b} yx),$$

$$(0 \xrightarrow{b} x)\theta_y = (0 \xrightarrow{b \otimes id_y} yx).$$

The quadruple (B, D, d, θ) defined as above is an E-system.

In the following lemmas, let $\mathcal{A}_{B \rightarrow D}$ and $\mathcal{A}_{B' \rightarrow D'}$ be Ann-categories associated to E-systems (B, D, d, θ) and (B', D', d', θ') , respectively.

Lemma 8. *Let $(f_1, f_0) : (B, D, d, \theta) \rightarrow (B', D', d', \theta')$ be a morphism of E-systems.*

i) *There is a functor $F : \mathcal{A}_{B \rightarrow D} \rightarrow \mathcal{A}_{B' \rightarrow D'}$ defined by*

$$F(x) = f_0(x), \quad F(b) = f_1(b), \quad x \in \text{Ob}(\mathcal{A}_{B \rightarrow D}), b \in \text{Mor}(\mathcal{A}_{B \rightarrow D}).$$

ii) *The functor F together with isomorphisms $\check{F}_{x,y} : F(x+y) \rightarrow Fx + Fy$, $\check{F}_{x,y} : F(xy) \rightarrow Fx Fy$ is an Ann-functor if $\check{F}_{x,y}$ and $\check{F}_{x,y}$ are constants in $\text{Ker} d'$ and for all $x, y \in D$ the following conditions hold:*

$$\theta'_{Fx}(\check{F}) = (\check{F})\theta'_{Fy} = \check{F}, \tag{16}$$

$$\theta'_{Fx}(\check{F}) = (\check{F})\theta'_{Fy} = \check{F} + \check{F}. \tag{17}$$

Then, we say that F is an Ann-functor of form (f_1, f_0) .

Proof. i) Every element $b \in B$ can be considered as a morphism $(0 \xrightarrow{b} db)$ in $\mathcal{A}_{B \rightarrow D}$. Then, $(F0 \xrightarrow{F(b)} F(db))$ is a morphism in $\mathcal{A}_{B' \rightarrow D'}$. By the construction of the Ann-category associated to an E-system, F is a functor.

ii) We define the natural isomorphisms

$$\check{F}_{x,y} : F(x+y) \rightarrow F(x) + F(y), \quad \tilde{F}_{x,y} : F(xy) \rightarrow F(x)F(y)$$

such that $F = (F, \check{F}, \tilde{F})$ becomes an Ann-functor. First we see that

$$F(x) + F(x') = F(x+x'),$$

so $d'(\check{F}_{x,x'}) = 0$. Analogously, $d'(\tilde{F}_{x,x'}) = 0$, thus

$$\check{F}_{x,x'}, \tilde{F}_{x,x'} \in \text{Ker } d' \subset C_{B'}. \quad (18)$$

Now, for two morphisms $(x \xrightarrow{b} y)$ and $(x' \xrightarrow{b'} y')$ in $\mathcal{A}_{B \rightarrow D}$, we have:

$$\bullet F(b \oplus b') = F(x+x' \xrightarrow{b+b'} y+y') = (f_0(x+x') \xrightarrow{f_1(b+b')} f_0(y+y')),$$

$$\begin{aligned} F(b) \oplus F(b') &= (f_0(x) \xrightarrow{f_1(b)} f_0(y)) \oplus (f_0(x') \xrightarrow{f_1(b')} f_0(y')) \\ &= (f_0(x) + f_0(x') \xrightarrow{f_1(b)+f_1(b')} f_0(y) + f_0(y')). \end{aligned}$$

Since f_1 is a ring homomorphism, one obtains

$$F(b \oplus b') = F(b) \oplus F(b'). \quad (19)$$

By (18) and (19), the commutativity of the diagram

$$\begin{array}{ccc} F(x+x') & \xrightarrow{\check{F}_{x,x'}} & F(x) + F(x') \\ \downarrow F(b \oplus b') & & \downarrow F(b) \oplus F(b') \\ F(y+y') & \xrightarrow{\check{F}_{y,y'}} & F(y) + F(y') \end{array} \quad (20)$$

follows from $\check{F}_{x,x'} = \check{F}_{y,y'}$.

$$\bullet F(b \otimes b') = F(xx' \xrightarrow{bb'+b\theta_{x'}+\theta_x b'} yy') = (f_0(xx') \xrightarrow{f_1(bb'+b\theta_{x'}+\theta_x b')} f_0(yy')),$$

$$\begin{aligned} F(b) \otimes F(b') &= (f_0(x) \xrightarrow{f_1(b)} f_0(y)) \otimes (f_0(x') \xrightarrow{f_1(b')} f_0(y')) \\ &= (f_0(x)f_0(x') \xrightarrow{f_1(b)f_1(b')+f_1(b)\theta'_{f_0(x')}+\theta'_{f_0(x)}f_1(b')} f_0(y)f_0(y')). \end{aligned}$$

By (12), $f_1(\theta_x b') = \theta'_{f_0(x)} f_1(b')$ and $f_1(b\theta_{x'}) = f_1(b)\theta'_{f_0(x')}$, hence

$$F(b \otimes b') = F(b) \otimes F(b'). \quad (21)$$

By (18) and (21), the commutativity of the diagram

$$\begin{array}{ccc} F(xx') & \xrightarrow{\tilde{F}_{x,x'}} & F(x)F(x') \\ F(b \otimes b') \downarrow & & \downarrow F(b) \otimes F(b') \\ F(yy') & \xrightarrow{\tilde{F}_{y,y'}} & F(y)F(y') \end{array} \quad (22)$$

follows from $\tilde{F}_{x,x'} = \tilde{F}_{y,y'}$. The equalities (16) and (17) come from the compatibility of (F, \tilde{F}) with the associativity constraint and the distributivity ones, respectively. \square

An Ann-functor F is *single* if $F(0) = 0'$, $F(1) = 1'$ and \check{F}, \tilde{F} are constants. Then we state the converse of Lemma 8.

Lemma 9. *Let $(F, \check{F}, \tilde{F}) : \mathcal{A}_{B \rightarrow D} \rightarrow \mathcal{A}_{B' \rightarrow D'}$ be a single Ann-functor. Then, there is a morphism of E-systems $(f_1, f_0) : (B \rightarrow D) \rightarrow (B' \rightarrow D')$, where*

$$f_1(b) = F(b), \quad f_0(x) = F(x),$$

for $b \in B, x \in D$.

Proof. Since $F(0) = 0'$, $F(1) = 1'$ and \check{F}, \tilde{F} are constants, it is easy to see that \check{F}, \tilde{F} are in $\text{Ker} d'$. By the determination of a morphism in $\mathcal{A}_{B' \rightarrow D'}$,

$$F(x + y) = F(x) + F(y), \quad F(xy) = F(x)F(y),$$

so f_0 is a ring homomorphism.

Since \check{F} is a constant in $\text{Ker} d'$, the commutative diagram (20) implies

$$F(b \oplus b') = F(b) \oplus F(b').$$

This means that $f_1(b + b') = f_1(b) + f_1(b')$.

Since \tilde{F} is a constant in $\text{Ker} d'$, the commutative diagram (22) implies

$$F(b \otimes b') = F(b) \otimes F(b').$$

By the definition of \otimes ,

$$f_1(bb') + f_1(b\theta_{x'}) + f_1(\theta_x b') = f_1(b)f_1(b') + f_1(b)\theta'_{f_0(x')} + \theta'_{f_0(x)} f_1(b'). \quad (23)$$

In this relation, taking $b = 0$ and then $b' = 0$ yield

$$f_1(\theta_x b') = \theta'_{f_0(x)} f_1(b'), \quad f_1(b\theta_{x'}) = f_1(b)\theta'_{f_0(x')}.$$

Thus, (12) holds. Then, the equation (23) turns into $f_1(bb') = f_1(b)f_1(b')$, that is, f_1 is a ring homomorphism. The rule (11) also holds. Indeed, for all morphisms $(x \xrightarrow{b} y)$ in $\mathcal{A}_{B \rightarrow D}$, $y = d(b) + x$. It follows that

$$f_0(y) = f_0(d(b) + x) = f_0(d(b)) + f_0(x).$$

Besides, $(f_0(x) \xrightarrow{f_1(b)} f_0(y))$ is a morphism in $\mathcal{A}_{B' \rightarrow D'}$, so

$$f_0(y) = d'(f_1(b)) + f_0(x).$$

Thus, $f_0(d(b)) = d'(f_1(b))$ for all $b \in B$. \square

Lemma 10. *Two Ann-functors $(F, \check{F}, \tilde{F}), (G, \check{G}, \tilde{G}) : \mathcal{A}_{B \rightarrow D} \rightarrow \mathcal{A}_{B' \rightarrow D'}$ of the same form are homotopic.*

Proof. Suppose that F and G are two Ann-functors of form (f_1, f_0) . By Lemma 8, \check{F}, \check{G} are constants. We prove that $\alpha = \check{G} - \check{F}$ is a homotopy between F and G . It is easy to check the naturality of α and the compatibility of α with the addition. Besides, α is compatible with the multiplication. In other words, the following diagram

$$\begin{array}{ccc} F(xy) & \xrightarrow{\tilde{F}} & F(x)F(y) \\ \alpha \downarrow & & \downarrow \alpha \otimes \alpha \\ G(xy) & \xrightarrow{\tilde{G}} & G(x)G(y) \end{array} \quad (24)$$

commutes. Indeed, by Lemma 8,

$$\begin{aligned} \tilde{G} - \tilde{F} &= (\theta'_{Fx}(\check{G}) - \check{G}) - (\theta'_{Fx}(\check{F}) - \check{F}) \\ &= \theta'_{Fx}(\alpha) - \alpha. \end{aligned}$$

Since $\alpha \in \text{Ker } d' \subset C_{B'}$, so

$$\begin{aligned} \alpha \otimes \alpha &= \alpha \cdot \alpha + (\alpha)\theta'_{Gy} + \theta'_{Gx}(\alpha) \\ &= (\alpha)\theta'_{Gy} + \theta'_{Gx}(\alpha). \end{aligned}$$

For $y = 0$, or $x = 0$ we have

$$\alpha \otimes \alpha = (\alpha)\theta'_{Gy} = \theta'_{Gx}(\alpha).$$

Thus,

$$\tilde{G} - \tilde{F} = \alpha \otimes \alpha - \alpha,$$

that is, (24) holds. \square

Two Ann-functors $(F, \check{F}, \tilde{F})$ and $(G, \check{G}, \tilde{G})$ are *strong homotopic* if they are homotopic and $F = G$. By Lemma (10), one obtains the following fact.

Corollary 11. *Two Ann-functors $F, G : \mathcal{A}_{B \rightarrow D} \rightarrow \mathcal{A}_{B' \rightarrow D'}$ are strong homotopic if and only if they are of the same form.*

We write **Annstr** for the category of strict Ann-categories and their single Ann-functors. We can define the *strong homotopy category* $Ho\mathbf{Annstr}$ to be the quotient category with the same objects, but morphisms are strong homotopy classes of single Ann-functors. We write $\text{Hom}_{\mathbf{Annstr}}[\mathcal{A}, \mathcal{A}']$ for the homsets of the homotopy category, that is,

$$\text{Hom}_{\mathbf{Annstr}}[\mathcal{A}, \mathcal{A}'] = \frac{\text{Hom}_{\mathbf{Annstr}}(\mathcal{A}, \mathcal{A}')}{\text{strong homotopies}}.$$

Denote **ESyst** the category of E-systems, we obtain the following result which is an extending of Theorem 1 [5]

Theorem 12 (Classification Theorem). *There exists an equivalence*

$$\begin{array}{ccc} \Phi : & \mathbf{ESyst} & \rightarrow Ho\mathbf{Annstr} \\ & (B \rightarrow D) & \mapsto \mathcal{A}_{B \rightarrow D} \\ & (f_1, f_0) & \mapsto [F] \end{array}$$

where $F(x) = f_0(x)$, $F(b) = f_1(b)$, for $x \in \text{Ob}\mathcal{A}$, $b \in \text{Mor}\mathcal{A}$.

Proof. By Corollary 11, the correspondence Φ on homsets,

$$\text{Hom}_{\mathbf{ESyst}}(B \rightarrow D, B' \rightarrow D') \rightarrow \text{Hom}_{\mathbf{Annstr}}[\mathcal{A}_{B \rightarrow D}, \mathcal{A}_{B' \rightarrow D'}],$$

is an injection. By Lemma 9, every single Ann-functor $F : \mathcal{A}_{B \rightarrow D} \rightarrow \mathcal{A}_{B' \rightarrow D'}$ determines a morphism of E-systems (f_1, f_0) , and clearly $\Phi(f_1, f_0) = [F]$, thus Φ is surjective on homsets.

Let $C_{\mathcal{A}}$ be an E-system associated to a strict Ann-category \mathcal{A} . By the construction of an Ann-category associated to an E-system, $\Phi(C_{\mathcal{A}}) = \mathcal{A}$ (rather than an isomorphism). Hence, Φ is an equivalence of categories. \square

5 Ring extensions of the type of an E-system

In this section we consider the ring extensions of the type of an E-system, which are analogous to the group extensions of the type of a crossed module [6].

Definition 5. Let (B, D, d, θ) be an E-system. A ring *extension* of B by Q of *type* $B \rightarrow D$ is a diagram of ring homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q \longrightarrow 0, \\ & & \parallel & & \downarrow \varepsilon & & \\ & & B & \xrightarrow{d} & D & & \end{array}$$

where the top row is exact, the quadruple (B, E, j, θ') is an E-system where θ' is given by the bimultiplication type, and the pair (id, ε) is a morphism of E-systems.

Two extensions of B by Q of type $B \xrightarrow{d} D$ are said to be *equivalent* if there is a morphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q \longrightarrow 0, \\ & & \parallel & & \downarrow \eta & & \parallel \\ 0 & \longrightarrow & B & \xrightarrow{j'} & E' & \xrightarrow{p'} & Q \longrightarrow 0, \end{array} \quad \begin{array}{c} E \xrightarrow{\varepsilon} D \\ E' \xrightarrow{\varepsilon'} D \end{array} \quad (25)$$

and $\varepsilon'\eta = \varepsilon$. Obviously, η is an isomorphism.

In the diagram

$$\mathcal{E} : \begin{array}{ccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q \longrightarrow 0, \\ & & \parallel & & \downarrow \varepsilon & & \vdots \psi \\ & & B & \xrightarrow{d} & D & \xrightarrow{q} & \text{Coker}d \end{array} \quad (26)$$

where q is a canonical projection, since the top row is exact and $q \circ \varepsilon \circ j = q \circ d = 0$, there is a ring homomorphism $\psi : Q \rightarrow \text{Coker}d$ such that the right hand side square commutes. Moreover, ψ depends only on the equivalence class of the extension \mathcal{E} . Our purpose is to study the set

$$\text{Ext}_{B \rightarrow D}(Q, B, \psi)$$

of equivalence classes of extensions of B by Q of type $B \rightarrow D$ inducing ψ . The results use the obstruction theory of Ann-functors

Let $\mathcal{A} = \mathcal{A}_{B \rightarrow D}$ be the Ann-category associated to an E-system $B \rightarrow D$. Clearly, $\pi_0 \mathcal{A} = \text{Coker}d$, $\pi_1 \mathcal{A} = \text{Ker}d$ and therefore the reduced Ann-category $S_{\mathcal{A}}$ is of form

$$S_{\mathcal{A}} = (\text{Coker}d, \text{Ker}d, k),$$

where $\bar{k} \in H_{Shu}^3(\text{Coker}d, \text{Ker}d)$ since \mathcal{A} and $S_{\mathcal{A}}$ are regular Ann-categories. The homomorphism $\psi : Q \rightarrow \text{Coker}d$ induces an *obstruction*,

$$\psi^* k \in Z_{Shu}^3(Q, \text{Ker}d), \quad (27)$$

which plays a fundamental role to state Theorem 13. This is the main result of this section, an extending of Theorem 5.2 [6]. Besides, a particular case of a regular E-system when $Q = \text{Coker}d$ and $\psi = id_{\text{Coker}d}$ is a ∂ -extension [4], so our result contains Theorem 4.4.2 [4].

Theorem 13. *Let (B, D, d, θ) be a regular E-system, $\psi : Q \rightarrow \text{Coker}d$ be a ring homomorphism. Then, the vanishing of $\overline{\psi^* k}$ in $H_{Shu}^3(Q, \text{Ker}d)$ is necessary and sufficient for there to exist a ring extension of B by Q of type $B \rightarrow D$ inducing ψ . Further, if $\overline{\psi^* k}$ vanishes then there is a bijection*

$$\text{Ext}_{B \rightarrow D}(Q, B, \psi) \leftrightarrow H_{Shu}^2(Q, \text{Ker}d).$$

The first assertion is based on the following lemmas.

Lemma 14. *For every Ann-functor $(F, \check{F}, \tilde{F}) : \text{Dis}Q \rightarrow \mathcal{A}$ there exists an extension \mathcal{E}_F of B by Q of type $B \rightarrow D$ inducing $\psi : Q \rightarrow \text{Coker}d$.*

Such extension \mathcal{E}_F is called an *associated* extension to Ann-functor F .

Proof. By Proposition 2, $(F, \check{F}, \tilde{F})$ induces an Ann-functor $K : \text{Dis}Q \rightarrow S_{\mathcal{A}}$ of type $(\psi, 0)$. Let $(H, \check{H}, \tilde{H}) : S_{\mathcal{A}} \rightarrow \mathcal{A}$ be a canonical Ann-functor defined by the stick (x_s, i_x) . By (2), we have

$$H(s) = x_s, H(s, b) = b, \check{H}_{s,r} = -i_{x_s+x_r}, \tilde{H}_{s,r} = -i_{x_s \cdot x_r}.$$

Also by Proposition 2, $(F, \check{F}, \tilde{F})$ is homotopic to the composition

$$\text{Dis}Q \xrightarrow{K} S_{\mathcal{A}} \xrightarrow{H} \mathcal{A}.$$

So one can choose $(F, \check{F}, \tilde{F})$ being this composition. By the determination of $\check{H}K$ and $\tilde{H}K$,

$$\check{F}_{u,v} = f(u, v) = f'(u, v) - i_{x_s+x_r}, \quad (28)$$

$$\tilde{F}_{u,v} = g(u, v) = g'(u, v) - i_{x_s \cdot x_r} \in B, \quad (29)$$

where $u, v \in Q, s = \psi(u), r = \psi(v), f'(u, v) = \check{K}_{u,v}, g'(u, v) = \tilde{K}_{u,v}$. By the compatibility of $(F, \check{F}, \tilde{F})$ with the strict constraints of $\text{Dis}Q$ and \mathcal{A} , the functions f and g are the “normal” ones satisfying

$$f(u, v+t) + f(v, t) - f(u, v) - f(u+v, t) = 0, \quad (30)$$

$$f(u, v) = f(v, u), \quad (31)$$

$$\theta_{Fu}g(v, t) - g(uv, t) + g(u, vt) - g(u, v)\theta_{Ft} = 0, \quad (32)$$

$$g(u, v+t) - g(u, v) - g(u, t) + \theta_{Fu}f(v, t) - f(uv, ut) = 0, \quad (33)$$

$$g(u+v, t) - g(u, t) - g(v, t) + f(u, v)\theta_{Ft} - f(ut, vt) = 0. \quad (34)$$

The function $\varphi : Q \rightarrow M_B$ defined by

$$\varphi(u) = \theta_{Fu} = \theta_{x_s} \quad (s = \psi(u))$$

satisfies the relations

$$\varphi(u) + \varphi(v) = \mu_{f(u,v)} + \varphi(u+v), \quad (35)$$

$$\varphi(u)\varphi(v) = \mu_{g(u,v)} + \varphi(uv). \quad (36)$$

We only prove the relation (35), the proof of (36) follows from (29) in the same way. Since $f'(u, v) = \check{K}_{u,v} \in \text{Ker}d$, then by Proposition 4, $f'(u, v) \in C_B$. By (28), one has $\mu_{f(u,v)} = \mu(-i_{x_s+x_r})$. Thus,

$$\begin{aligned} \varphi(u) + \varphi(v) &= \theta_{x_s} + \theta_{x_r} = \theta_{x_s+x_r} \\ &= \theta[d(-i_{x_s+x_r}) + x_{s+r}] = \theta[d(-i_{x_s+x_r})] + \theta_{x_{s+r}} \\ &= \mu(-i_{x_s+x_r}) + \varphi(u+v) \stackrel{(28)}{=} \mu_{f(u,v)} + \varphi(u+v). \end{aligned}$$

Since the family of functions (φ, f, g) satisfies the relations (30) - (36), we have a crossed product $E_0 = [B, \varphi, f, g, Q]$, that means $E_0 = B \times Q$, and two operations are

$$(b, u) + (b', u') = (b + b' + f(u, u'), u + u'),$$

$$(b, u) \cdot (b', u') = (b \cdot b' + b\varphi(u') + \varphi(u)b' + g(u, u'), uu').$$

The set E_0 satisfies the axioms of a ring, in which note that the associativity for the multiplication in E_0 holds if and only if the E-system $B \rightarrow D$ is regular. Indeed, one can calculate the triple products as follows:

$$\begin{aligned} [(b, u)(b', u')](b'', u'') &= ((bb')b'' + b\varphi(u')\varphi(u'') + [\varphi(u)b']\varphi(u'') \\ &\quad + g(u, u')\varphi(u'') + \varphi(uu')b'' + g(uu', u''), (uu')u''), \end{aligned}$$

$$\begin{aligned} (b, u)[(b', u')(b'', u'')] &= (b(b'b'') + b\varphi(u'u'') + \varphi(u)[b'\varphi(u'')]) \\ &\quad + \varphi(u)\varphi(u')b'' + \varphi(u)g(u, u') + g(u, u'u''), u(u'u'')), \end{aligned}$$

By (32), (36), associative law for the multiplication in B, Q , and commutative law for the addition in B , especially by the relation (8), $[\varphi(u)b']\varphi(u'') = \varphi(u)[b'\varphi(u'')]$, we get the associative law for product in E_0 . Then, there is an exact sequence of ring homomorphisms

$$\mathcal{E}_F : 0 \rightarrow B \xrightarrow{j_0} E_0 \xrightarrow{p_0} Q \rightarrow 0,$$

where $j_0(b) = (b, 0)$; $p_0(b, u) = u$, $b \in B, u \in Q$. Since $j_0(B)$ is a two-sided ideal in E_0 , $B \xrightarrow{j_0} E_0$ is an E-system, where $\theta_0 : E_0 \rightarrow M_B$ is given by the bimultiplication type.

We define a ring homomorphism $\varepsilon : E_0 \rightarrow D$ by

$$\varepsilon(b, u) = db + x_{\psi(u)}, (b, u) \in E_0,$$

where $x_{\psi(u)}$ is a representative of u in D . We show that the pair (id_B, ε) satisfies the rules (11), (12). Clearly, $\varepsilon \circ j_0 = d$. Besides, for all $(b, u) \in E_0, c \in B$,

$$\theta_0(b, u)(c) = j_0^{-1}[(b, u)(c, 0)] = bc + \varphi(u)c,$$

$$\theta_{\varepsilon(b, u)}(c) = \theta_{db+x_{\psi(u)}}c = bc + \varphi(u)c.$$

Thus, $\theta_0(b, u)(c) = \theta_{\varepsilon(b, u)}(c)$. Analogously, $c\theta_0(b, u) = c\theta_{\varepsilon(b, u)}$. So (id_B, ε) is a morphism of E-systems, that is, one has an extension (26), where E is replaced by E_0 .

For all $u \in Q$ we have $q\varepsilon(0, u) = q(x_{\psi(u)}) = \psi(u)$, then the extension \mathcal{E}_F induces $\psi : Q \rightarrow \text{Coker } d$. \square

The proof of Theorem 13

Proof. Let us recall that \mathcal{A} is the Ann-category associated to the regular E-system $B \xrightarrow{d} D$. Then, its reduced Ann-category is $S_{\mathcal{A}} = (\text{Coker}d, \text{Ker}d, k)$, where $k \in Z_{Shu}^3(\text{Coker}d, \text{Ker}d)$. The pair

$$(\psi, 0) : (Q, 0, 0) \rightarrow (\text{Coker}d, \text{Ker}d, k)$$

has $-\psi^*k$ as an obstruction. By the assumption, $\overline{\psi^*k} = 0$, hence by Proposition 3 the pair $(\psi, 0)$ determines an Ann-functor $(\Psi, \check{\Psi}, \tilde{\Psi}) : \text{Dis}Q \rightarrow S_{\mathcal{A}}$. Then the composition of $(\Psi, \check{\Psi}, \tilde{\Psi})$ and $(H, \check{H}, \tilde{H}) : S_{\mathcal{A}} \rightarrow \mathcal{A}$ is an Ann-functor $(F, \check{F}, \tilde{F}) : \text{Dis}Q \rightarrow \mathcal{A}$, and by Lemma 14 we obtain an associated extension \mathcal{E}_F .

Conversely, suppose that there is an extension as in the diagram (26). Let \mathcal{A}' be the Ann-category associated to the E-system $B \rightarrow E$. By Proposition 2, there is an Ann-functor $F : \mathcal{A}' \rightarrow \mathcal{A}$. Since the reduced Ann-category of \mathcal{A}' is $\text{Dis}Q$, so by Proposition 2, F induces an Ann-functor of type $(\psi, 0)$ from $\text{Dis}Q$ to $(\text{Coker}d, \text{Ker}d, k)$. Now, by Proposition 3, the obstruction of the pair $(\psi, 0)$ must vanish in $H_{Shu}^3(Q, \text{Ker}d)$, that is, $\overline{\psi^*k} = 0$. \square

The final assertion of Theorem 13 follows from the next theorem.

Theorem 15 (Schreier Theory for ring extensions of the type of an E-system). *There is a bijection*

$$\Omega : \text{Hom}_{(\psi, 0)}^{Ann}[\text{Dis}Q, \mathcal{A}] \rightarrow \text{Ext}_{B \rightarrow D}(Q, B, \psi).$$

Proof. Step 1: The Ann-functors $(F, \check{F}, \tilde{F})$, $(F', \check{F}', \tilde{F}')$ are homotopic if and only if their corresponding associated extensions $\mathcal{E}_F, \mathcal{E}_{F'}$ are equivalent.

Let two Ann-functors $F, F' : \text{Dis}Q \rightarrow \mathcal{A}$ be homotopic by a homotopy $\alpha : F \rightarrow F'$. Then, by the definition of an Ann-morphism, the following diagrams commute

$$\begin{array}{ccc} F(u+v) & \xrightarrow{\check{F}_{u,v}} & F(u) + F(v) \\ \alpha_{u+v} \downarrow & & \downarrow \alpha_u + \alpha_v \\ F'(u+v) & \xrightarrow{\check{F}'_{u,v}} & F'(u) + F'(v), \end{array} \quad \begin{array}{ccc} F(uv) & \xrightarrow{\tilde{F}_{u,v}} & F(u)F(v) \\ \alpha_{uv} \downarrow & & \downarrow \alpha_u \otimes \alpha_v \\ F'(uv) & \xrightarrow{\tilde{F}'_{u,v}} & F'(u)F'(v). \end{array}$$

By the definition of the operation \otimes on \mathcal{A} ,

$$\alpha_u \otimes \alpha_v = \alpha_u \alpha_v + \alpha_u \theta_{Fv} + \theta_{Fu} \alpha_v.$$

Then, since $f(u, v) = \check{F}_{u,v}$, $f'(u, v) = \check{F}'_{u,v}$, $g(u, v) = \tilde{F}_{u,v}$, $g'(u, v) = \tilde{F}'_{u,v}$, we have

$$f'(u, v) - f(u, v) = \alpha_u - \alpha_{u+v} + \alpha_v, \quad (37)$$

$$g'(u, v) - g(u, v) = \alpha_u \alpha_v + \alpha_u \theta_{Fv} + \theta_{Fu} \alpha_v - \alpha_{uv}. \quad (38)$$

Now, we set

$$\begin{aligned}\alpha^* : E_F &\rightarrow E_{F'} \\ (b, u) &\mapsto (b - \alpha_u, u).\end{aligned}$$

Note that $\theta_{F'u} = \mu_{\alpha_u} + \theta_{Fu}$, and by the relations (37), (38), the correspondence α^* is an isomorphism. Besides, the diagram (25) commutes in which E and E' are replaced by E_F and $E_{F'}$, respectively.

Finally, $\varepsilon' \alpha^* = \varepsilon$. Indeed, since $\alpha : F \rightarrow F'$ is a homotopy, then $Fu = x_{\psi(u)} = F'u$. Thus $x_{\psi(u)} = d(\alpha_u) + x_{\psi(u)}$, or $d(\alpha_u) = 0$. Hence,

$$\begin{aligned}\varepsilon' \alpha^*(b, u) &= \varepsilon'(b - \alpha_u, u) = d(b - \alpha_u) + x_{\psi(u)} \\ &= d(b) - d(\alpha_u) + x_{\psi(u)} = d(b) + x_{\psi(u)} = \varepsilon(b, u).\end{aligned}$$

That means two extensions \mathcal{E}_F and $\mathcal{E}_{F'}$ are equivalent.

Conversely, if \mathcal{E}_F and $\mathcal{E}_{F'}$ are equivalent, there exists a ring isomorphism $(b, u) \mapsto (b - \alpha_u, u)$. Then, we have a homotopy $\alpha : F \rightarrow F'$ by retracing our steps.

Step 2: Ω is a surjection.

Let \mathcal{E} be an extension E of B by Q of type (B, D, d, θ) inducing $\psi : Q \rightarrow \text{Coker } d$ (see the commutative diagram (26)). We prove that \mathcal{E} is equivalent to an extension \mathcal{E}_F which is associated to an Ann-functor $(F, \check{F}, \tilde{F}) : \text{Dis}Q \rightarrow \mathcal{A}$.

Let $\mathcal{A}' = \mathcal{A}_{B \rightarrow E}$ be the Ann-category associated to the E-system (B, E, j, θ') . By Lemma 8, the pair (id_B, ε) in the diagram (26) determines a single Ann-functor $(K, \check{K}, \tilde{K}) : \mathcal{A}' \rightarrow \mathcal{A}$.

Since $\pi_0 \mathcal{A}' = Q, \pi_1 \mathcal{A}' = 0$, the reduced Ann-category $S_{\mathcal{A}'}$ is nothing else but the Ann-category $\text{Dis}Q$. Choose a stick $(e_u, i_e), e \in E, u \in Q$, of \mathcal{A}' (that is, $\{e_u\}$ is a representative of Q in E). By (2), the canonical Ann-functor $(H', \check{H}', \tilde{H}') : \text{Dis}Q \rightarrow \mathcal{A}'$ is given by

$$H'(u) = e_u, \check{H}'_{u,v} = -i_{e_u + e_v} = g'(u, v), \tilde{H}'_{u,v} = -i_{e_u \cdot e_v} = h'(u, v).$$

The composition $F = K \circ H'$ is an Ann-functor $\text{Dis}Q \rightarrow \mathcal{A}$, where

$$F(u) = \varepsilon(e_u), \check{F}_{u,v} = \check{H}'_{u,v} = g'(u, v), \tilde{F}_{u,v} = \tilde{H}'_{u,v} = h'(u, v).$$

According to the proof of Theorem 13, we construct an extension \mathcal{E}_F of the crossed product $E_0 = [B, \varphi, g', h', Q]$ which is associated to $(F, \check{F}, \tilde{F})$.

We now prove that \mathcal{E} and \mathcal{E}_F are equivalent, that is, there is a commutative diagram

$$\begin{array}{ccccccc} \mathcal{E}_F : & 0 & \longrightarrow & B & \xrightarrow{j_0} & E_0 & \xrightarrow{p_0} & Q & \longrightarrow & 0 \\ & & & \parallel & & \downarrow \eta & & \parallel & & \\ \mathcal{E} : & 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q & \longrightarrow & 0 \end{array} \quad \begin{array}{c} E_0 \xrightarrow{\varepsilon_0} D \\ \\ E \xrightarrow{\varepsilon} D \end{array}$$

and $\varepsilon\eta = \varepsilon_0$.

Indeed, since every element of E can be written uniquely as $b + e_u, b \in B$, we can define a map

$$\eta : E_0 \rightarrow E, (b, u) \mapsto b + e_u.$$

We next verify that η is a ring isomorphism. The representatives e_u have the following properties

$$\varphi(u)c = \theta'_{e_u} c, c\varphi(u) = c\theta'_{e_u}, c \in B, \quad (39)$$

$$e_u + e_v = -i_{e_u+e_v} + e_{u+v} = g'(u, v) + e_{u+v}, \quad (40)$$

$$e_u \cdot e_v = -i_{e_u \cdot e_v} + e_{u \cdot v} = h'(u, v) + e_{uv}. \quad (41)$$

(The relation (39) holds since the pair (id_B, ε) is a morphism of E-systems. The relations (40), (41) hold thanks to the definition of a morphism in \mathcal{A}' .) Now, we have

$$\begin{aligned} \eta[(b, u) + (c, v)] &= \eta(b + c + g'(u, v), u + v) = b + c + g'(u, v) + e_{u+v} \\ &\stackrel{(40)}{=} b + c + e_u + e_v = (b + e_u) + (c + e_v) = \eta(b, u) + \eta(c, v). \end{aligned}$$

$$\begin{aligned} \eta[(b, u)(c, v)] &= \eta(bc + b\varphi(v) + \varphi(u)c + h'(u, v), uv) \\ &= bc + b\varphi(v) + \varphi(u)c + h'(u, v) + e_{uv} \\ &\stackrel{(39), (41)}{=} bc + b\theta'_{e_v} + \theta'_{e_u}c + e_u e_v \\ &= bc + b \cdot e_v + e_u \cdot c + e_u \cdot e_v \\ &= (b + e_u) \cdot (c + e_v) = \eta(b, u) \cdot \eta(c, v). \end{aligned}$$

Finally, choose the representative e_u such that $\varepsilon(e_u) = x_{\psi(u)}$ (since it follows from (26) that $q(\varepsilon(e_u)) = \psi p(e_u) = \psi(u)$). Thus,

$$\varepsilon\eta(b, u) = \varepsilon(b + e_u) = \varepsilon(b) + \varepsilon(e_u) = d(b) + x_{\psi(u)} = \varepsilon_0(b, u),$$

that is, \mathcal{E} and \mathcal{E}_F are equivalent. \square

Now, the bijection mentioned in Theorem 13 is obtained as follows. Note that there is a natural bijection

$$\text{Hom}[\text{Dis}Q, \mathcal{A}] \leftrightarrow \text{Hom}[\text{Dis}Q, S_{\mathcal{A}}].$$

Then, since $\pi_0(\text{Dis}Q) = Q$ and $\pi_1(S_{\mathcal{A}}) = \text{Ker}d$, Theorem 15 and Theorem 3 imply

$$\text{Ext}_{B \rightarrow D}(Q, B, \psi) \leftrightarrow H_{Shu}^2(Q, \text{Ker}d).$$

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